How I Think About Math
Part I: Linear Algebra

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Chapter 1: Relations

1. Relations
   - Labels
   - Composing
   - Joining
   - Inverting
   - Commuting

2. Linearity
   - Fields
     - “Linear” defined
   - Vectors
   - Matrices
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   - Fundamental Theorem of Linear Algebra
   - CP decomposition
Relations are a generalization of functions; they’re actually more like constraints. Here’s an example:

\[ x \rightarrow 2 \cdot \rightarrow y \]
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\[ x \rightarrow 2 \cdot y \]

This might be more familiar to you as the equation:

\[ y(x) = 2 \cdot x \]
Relations are a generalization of functions; they’re actually more like constraints. Here’s an example:

\[
3 \rightarrow 2 \cdot 6
\]

This might be more familiar to you as the equation:

\[
6 = 2 \cdot 3
\]
Relations are a generalization of functions; they’re actually more like constraints. Here’s an example:

\[
2.5 \rightarrow 2 \cdot 5
\]

This might be more familiar to you as the equation:

\[
5 = 2 \cdot 2.5
\]
Relations are a generalization of functions; they’re actually more like constraints. Here’s an example:

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\end{align*}
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Really, the directional annotations on the arrows are just that: annotations.

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Analogously, writing $y(x)$ is just politics: “$x$ gets to tell $y$ what to do!”
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![Diagram](image)

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Relations are a generalization of functions; they’re actually more like constraints. Here’s an example:

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A simpler relation

\[ x \quad \longrightarrow \quad y \]
A simpler relation

You might better know this relation as

\[ y = x \]
You might better know this relation as

3 = 3
A simpler relation

You might better know this relation as

\[ 2 = 2 \]
You might better know this relation as

\[ 0 = 0 \]
A simpler relation

\[ x \quad \overset{\text{Y}}{\leftrightarrow} \quad y \]

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• Like the arguments of a subroutine, the labels of a relation are just a convenient “interface” for connecting it to a context or environment.
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If a label isn’t serving that purpose, we can remove it.
Composing two relations
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Sticking relations together like this will always give you a relation.
What does this mean?

\[ x \rightarrow y \]

\[ \downarrow \]

\[ z \]
What does this mean?

You could think of it as:
What does *this* mean?

\[ x \quad \downarrow \quad y \]

\[ z \]

You could think of it as:

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You could think of it as:

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\[ x \rightarrow y \]

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What does *this* mean?

You could think of it as:

\[
\begin{align*}
x &= y \\
x &= z \\
y &= x \\
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They’re all the same! But with complex joins, this is easier to see in pictures.
Joined Relations

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  y &= x \\
  x &= z \quad \text{or} \quad y &= z \\
  z &= y
\end{align*}
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They’re all the same! But with complex joins, this is easier to see in pictures. Relations with more than two “sides” (like this) are sometimes called

**systems of equations.**

But I find a single 3-sided relation more intuitive than a “system” of two equations.
An example inverse

Let’s write “multiplication by 0.5 is the inverse of multiplication by 2.”

\[
\begin{array}{c}
\text{let} \quad 2 \cdot y = x \\
\text{let} \quad 0.5 \cdot x = y \\
\end{array}
\]
Let’s write “multiplication by 0.5 is the inverse of multiplication by 2.”

\[
x \quad 2 \cdot \quad y
\]

\[
y \quad 0.5 \cdot \quad x
\]

Note: This is like the system of equations

\[
y = 2 \cdot x
\]

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\text{Of course, } x = x \text{ and } y = y, \text{ so we can join those relations in.}
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Let’s write “multiplication by 0.5 is the inverse of multiplication by 2.”

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- Of course, $x = x$ and $y = y$, so we can join those relations in.
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- The meaning is still imprecise. Even this is valid if $x$ happens to be 0.
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Let’s write “$A^{-1}$ is the inverse of $A$ over $\mathbb{R}$.”

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If we can reverse the order of two operators and get equal results, we say that they **commute**.
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What is “Linear”?
Algebra
davidad

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