How I Think About Math
Part I: Linear Algebra

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1 Relations
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- Joining
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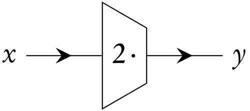
A simple relation

Relations are a generalization of functions; they’re actually more like constraints. Here’s an example:



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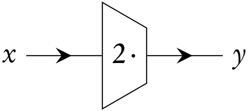

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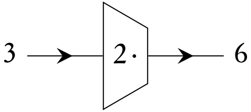

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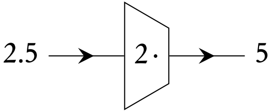

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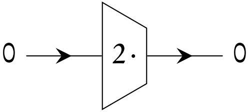

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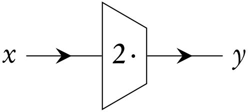

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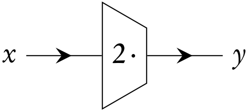

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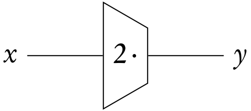

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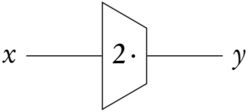

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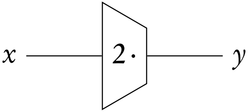

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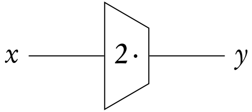

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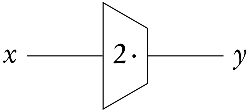

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$$x \;\text{———}\; y$$

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You might better know this relation as

$$y = x$$

A simpler relation

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$$2 \;\text{———}\; 2$$

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$$0 \;\text{———}\; 0$$

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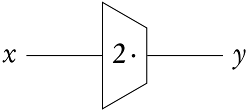
A simpler relation

$$x \longrightarrow y$$

You might better know this relation as

$$y = x$$

A word about labels



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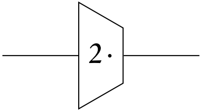
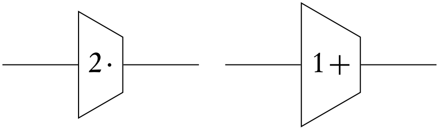


- Like the arguments of a subroutine, the labels of a relation are just a convenient "interface" for connecting it to a context or environment.
- If a label isn't serving that purpose, we can remove it.

Composing two relations



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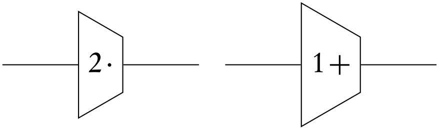

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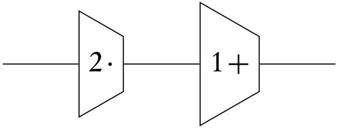

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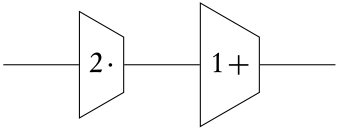

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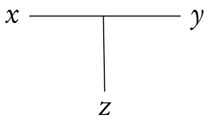
Sticking relations together like this will always give you a relation.

Joined Relations

What does *this* mean?



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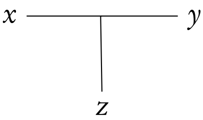

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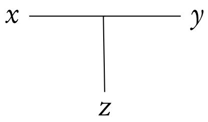

You could think of it as:

$$x = y$$

$$x = z$$

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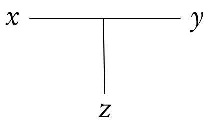

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$$\text{or}$$
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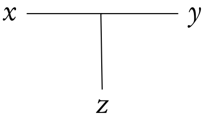

You could think of it as:

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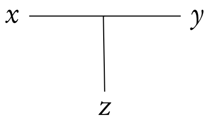

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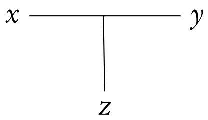

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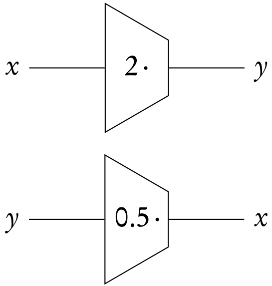
Relations with more than two “sides” (like this) are sometimes called

systems of equations.

But I find a single 3-sided relation more intuitive than a “system” of two equations.

An example inverse

Let’s write “multiplication by 0.5 is the inverse of multiplication by 2.”



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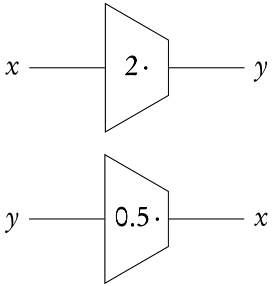

Note: This is like the system of equations

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$$x = 0.5 \cdot y$$

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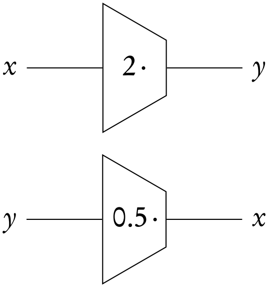

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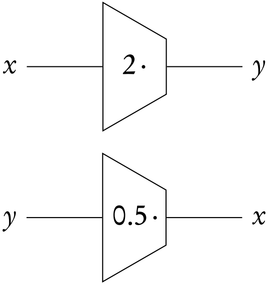

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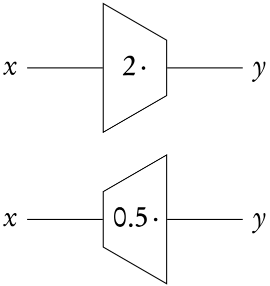

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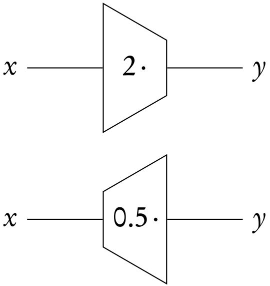

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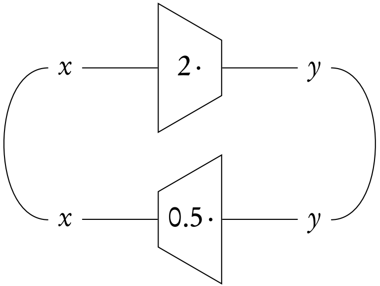

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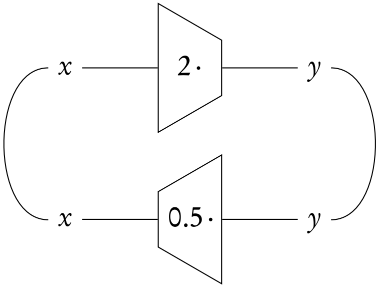

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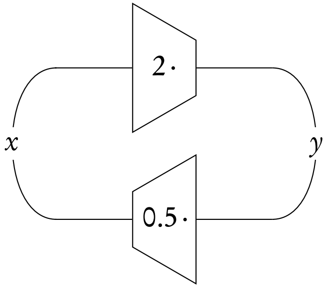

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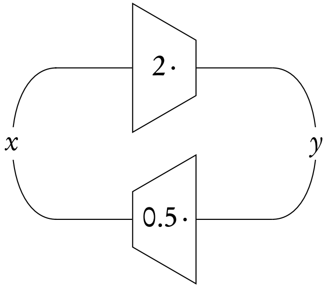

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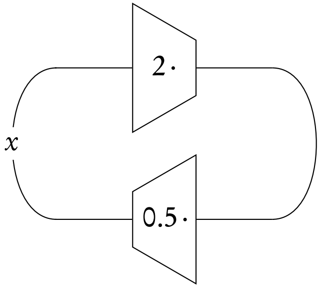

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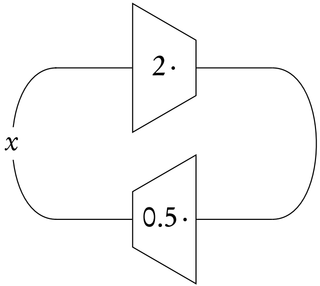

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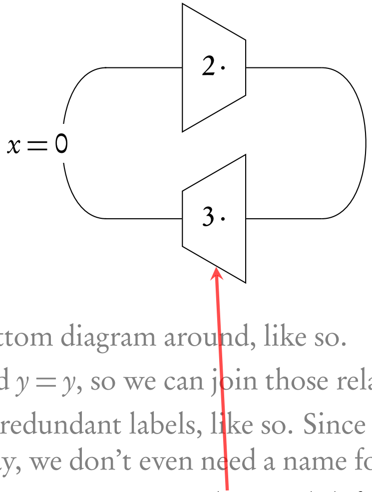

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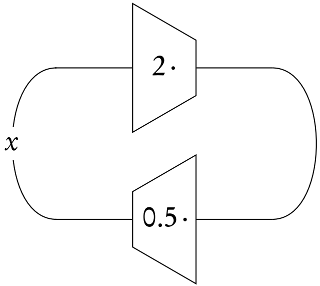

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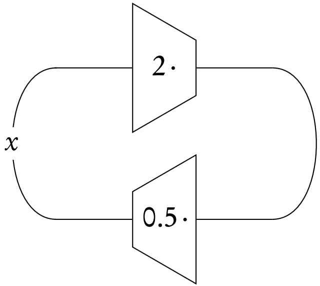

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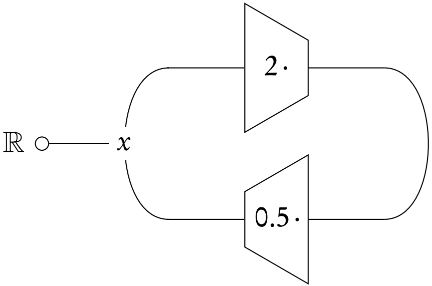

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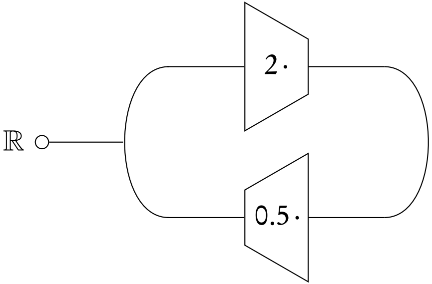

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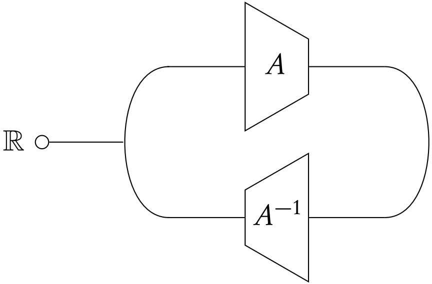
An example inverse

Let's write "A^{-1} is the inverse of A over $\mathbb{R}$."



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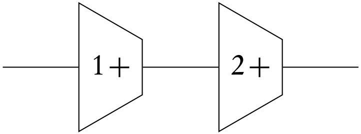

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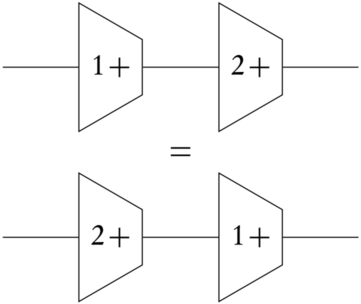
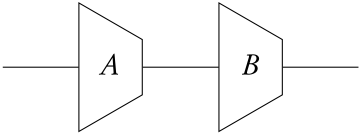


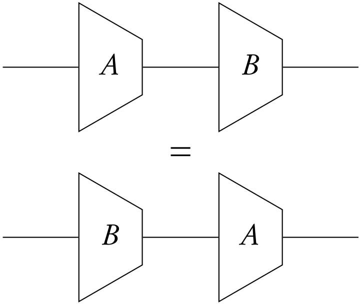
Commutativity

If we can reverse the order of two operators A and B and get equal results, we say that A and B **commute**.



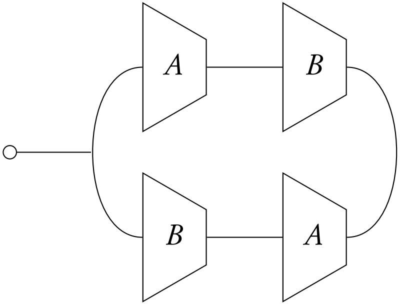
Commutativity

If we can reverse the order of two operators A and B and get equal results, we say that A and B **commute**.



Commutativity

If we can reverse the order of two operators A and B and get equal results, we say that A and B **commute.** We can express "A and B commute" like this:



What is “Linear”?